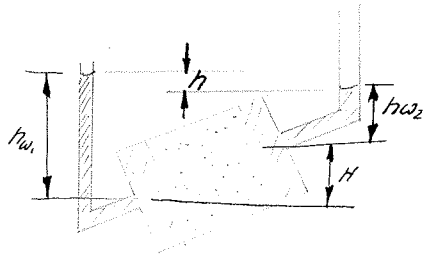
Seepage through Porous Media

Darcy's Law



h_w = Piezometric head

$p = h_w \gamma_w$ = piezometric pressure
= total pressure
= actual fluid pressure
w.r.t atmospheric

h = hydraulic head = head required to cause flow
= head lost by friction thru. media.

$u = h\gamma_w$ = excess pressure.
= an actual potential as opposed to $h_w \gamma_w$ which is a total pressure and not necessarily a potential.

Pressure gradient = $\frac{h\gamma_w}{L} = i_p$

Hydraulic gradient = $\frac{h}{L} = \frac{i_p}{\gamma_w} = i$.

Darcy's law states:

$$V = ki$$

where V = superficial velocity of discharge
i = hydraulic gradient

The seepage velocity = $\frac{Q}{Av} = \frac{Q}{nA} = V_s$

superficial velocity = $\frac{Q}{A} = V$ { Always work with V & can determine V_s if required }

Hence $V_s = \frac{V}{n}$.

Rigorous Statement of Darcy's law.

$$V = \frac{K}{\mu} i_p = \frac{K}{\mu} \gamma_w i$$

where K = True permeability - a function of geometry only.
Has units $(\text{Length})^2$.

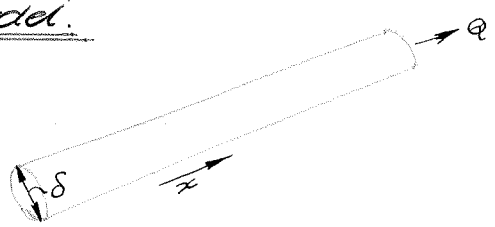
μ = Viscosity

$k = \frac{K \cdot \gamma_w}{\mu}$ = "hydraulic conductivity"
"coefficient of permeability"

Physics of permeability

In this approach K is calculated using geometry eg crack pattern in fractured rock.
The value of K is calculated using models.

a: The straight capillary model.



The steady state flow in such a tube is given by

$$Q = \frac{\pi \delta^4}{128 \mu} \frac{d\phi}{dx} \qquad \text{where } \phi = h_w \gamma_w.$$

Consider the media as being a bundle of such small tubes.
there being N capillaries/unit area.

$$\therefore \quad q = \frac{N \pi \delta^4}{128 \mu} \frac{d\phi}{dx} \quad \text{per unit area.}$$

From Darcy we have that $V = \frac{K}{\mu} i\rho$

$$\text{Thus} \quad K = \frac{N \pi \delta^4}{128} \qquad \left(N = \frac{1}{L^2}\right)$$

$$\text{But} \quad n = \frac{V_v}{1.0} = \frac{N \pi \delta^2}{4}$$

$$\therefore \quad K = \frac{n \delta^2}{32.}$$

We thus have a formula for predicting K and although it is obviously not precise it is to be noted that

Hazens formula : $K = C D_{10}^2$

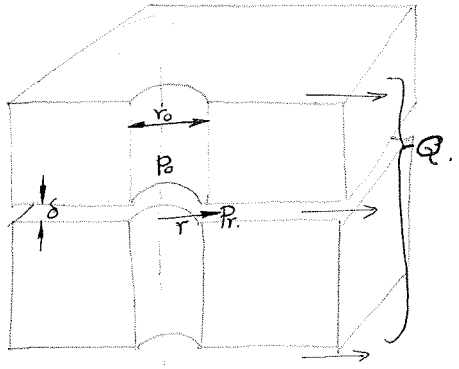
$C = 100 - 150$

This model thus predicts the structure of Hazens formula.

b: Model related to fractured media.

Consider N fissures per unit length across a tube.

N : fissures/unit length over length : L.



From fluid mechanics we have

$$Q = \frac{NL\,(p_o - p_r)\,\delta^3\,\pi}{6\mu \log_e r/r_o}$$

For a uniform porous medium $Q = \frac{2\pi L k}{\gamma_w} \times \frac{p_o - p_r}{\log_e r/r_o}$

Thus the equivalent permeability of the fissured structure

$$k = \frac{N\delta^3\gamma_w}{12\mu}$$

and since $k = \frac{K\gamma_w}{\mu}$

we have $K = \frac{N\delta^3}{12}$ which in practice $K = C\frac{N\delta^3}{12}$

where C = equivalent to a roughness factor about which very little is known.

NOTE : ① Consider $k_{equiv} = 10^{-4}$ cm/sec. $N = 1.0$

$$\therefore \quad \delta = \sqrt[3]{\frac{12\mu \times 10^{-4}}{\gamma_w}} = 7.5 \times 10^{-3} \text{ cm.}$$

Thus one narrow fissure per ft makes a great diff to k.

② Average cement particle $= 1.0 \times 10^{-2}$ cm.
Hence grouting is a waste of time & $

Influence of pressure on permeability

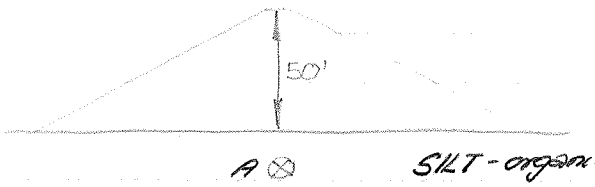
a/ Soils

In general $+\Delta p \rightarrow -\Delta e \rightarrow -\Delta k$.

i/ In sandy soils, effect is small
ii/ Silts & silty clays – extremely significant
iii/ Fat clays – significant but not relevant.

If there is loosely packed organic material a 2 orders of magnitude in $p' \longrightarrow$ 4 orders of magnitude change in k.

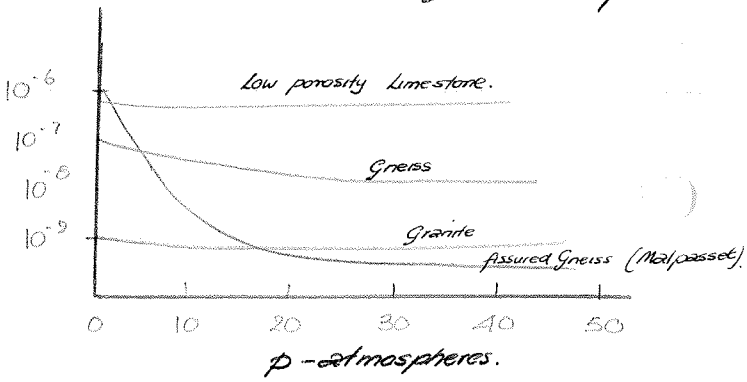
At A the high pressure leads to a reduction in permeability sufficiently large to produce a self made cut-off of considerable significance



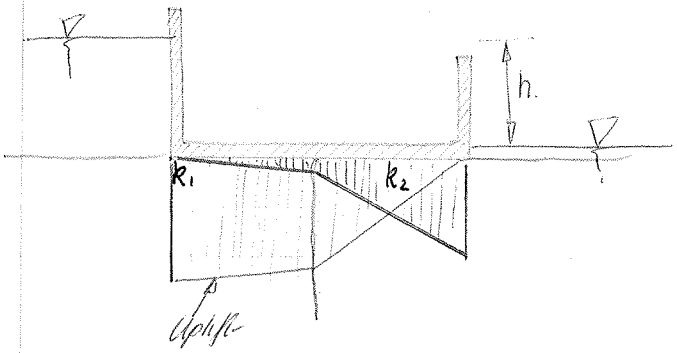
b/ Rocks.

Seemingly intact rocks are often microfissured.

When the bulk permeability depends on fissures a slight opening of the fissures leads to a large change in permeability.



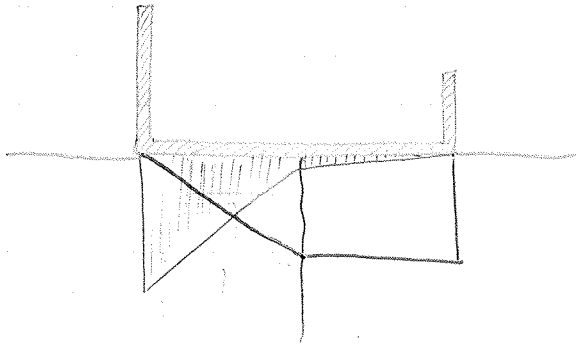
Because of pressure effects the pore pressure distribution in a fractured rock can be rather mobile.



$k_1 \gg k_2$.

$\therefore \Delta H_1 \ll \Delta H_2$

$\therefore$ Full up lift pressures on LHS.



$k_1 \ll k_2$

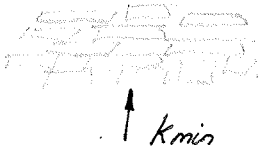
$\Delta H_1 \gg \Delta H_2$

Full uplift on RHS but reduced

Effect of Soil Structure on Permeability

1/ Anisotropy

There is a degree of preferred orientation in all soils. From experiments with "perfectally oriented" kaolin it has been shown that



$\frac{k_{max}}{k_{min}} \not> 3$

In normal soils $k_{max}/k_{min} \ll 3.0$.

2/ Non-homogeneity

This is the cause of great differences between bulk k_{horiz} & k_{vert}.

It is possible to average out values for k_h & k_v and then perform the analysis as for an anisotropic medium.

$k = 1 \times 10^{-6}$

The effect of the thin gravel layers changed an expected 3·0 cu ft/day to 1×10^{6} cu ft/day.

NB: If local G.W.T responds quickly to changes in river level etc. then one must suspect such layers.

Equivalent k of a non-homogenous mass of soil

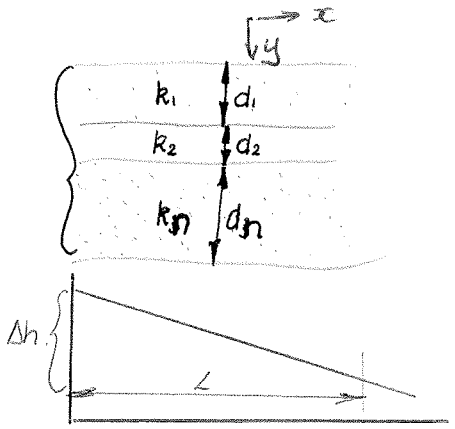
HORIZ.

$$q_x = \sum_{1}^{n} k_m d_m \frac{\Delta h}{L}$$

$$v_x = \frac{\sum_{1}^{n} k_m d_m \, \Delta H/L}{d}$$

$$d = \sum_{m=1}^{n} d_m$$

$$\therefore \quad k_x = \text{equ.} = \frac{\sum_{1}^{n} k_m d_m}{d}$$



VERT.

$$q_1 = q_2 = \cdots = q_n$$
$$v_1 = v_2 = = v_n$$
$$k_1 i_1 = k_2 i_2 = \cdots = k_n i_n$$

Total head loss $= id = i_1 d_1 + i_2 d_2 + \cdots + i_n d_n$

$$= \sum_{1}^{n} i_m d_m$$

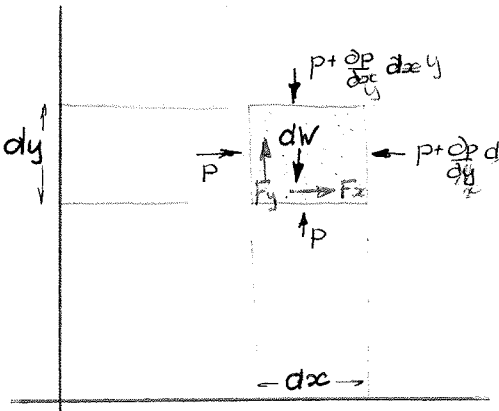
$$\therefore \quad i = \frac{\sum_{1}^{n} i_m d_m}{d} = \sum_{1}^{n} \frac{v_m}{k_m} \cdot \frac{d_m}{d}$$

$$= \frac{v}{d} \sum_{1}^{n} \frac{d_m}{k_m}$$

$$\therefore \quad k_v = \frac{d}{\sum_{1}^{n} \frac{d_m}{k_m}}$$

It can be shown that $k_x \gg k_v$ always.

Equation of Motion of Water.



dW = weight of water in element $-1/L^2$.
F_x & F_y = Drag Forces. (body forces act throughout water).
p = total water pressure.

Vertical Equilibrium

$$0 = \{p \cdot n \cdot dx\} - \left\{\left(p + \frac{\partial p}{\partial y} dy\right) \cdot n \cdot dx\right\} + F_y \cdot n \cdot dx \cdot dy - \gamma_w \cdot n \cdot dx \cdot dy$$

$$0 = \frac{\partial p}{\partial y} dy - F_y \cdot dy + \gamma_w \cdot dy$$

Thus $$\frac{\partial p}{\partial y} - F_y + \gamma_w = 0. \quad \text{—— (1)}$$

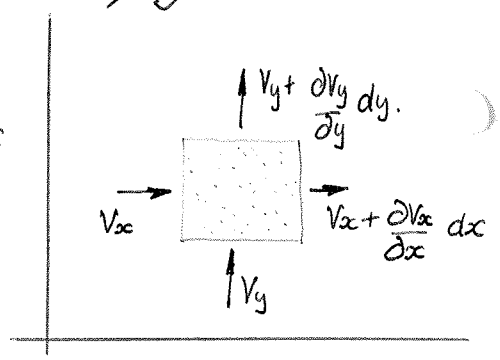
Horizontal Equilibrium.

$$\frac{\partial p}{\partial x} - F_x = 0 \quad \text{—— (2)}$$

It is the drag forces which give rise to the pressure gradients.

Equation of continuity:

Assumes
1/ Water is incompressible
2/ Arrangement of grains remains same
3/ Steady state seepage.



Volume in = $V_x \cdot dy + V_y \cdot dx$
Volume out = $\left\{V_x + \frac{\partial V_x}{\partial x} dx\right\} dy + \left\{V_y + \frac{\partial V_y}{\partial y} dy\right\} dx$

Thus by the above assumptions

$$\frac{\partial V_x}{\partial x} \cdot dx \cdot dy + \frac{\partial V_y}{\partial y} dy \cdot dx = 0$$

$$\therefore \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \quad \text{—— (3)}$$

V_x, V_y = Volumes/unit area.

Darcys Law:

$$\left.\begin{aligned} F(x) &= -\frac{\gamma_w}{k} V_x \\ F(y) &= -\frac{\gamma_w}{k} V_y. \end{aligned}\right\} \text{—— (4)}$$

Substituting into eq^n 1&2

$$\frac{\partial p}{\partial y} + \frac{\gamma_w}{k} V_y + \gamma_w = 0$$

$$\therefore \quad V_y = -\frac{k}{\gamma_w}\left\{\frac{\partial p}{\partial y} + \gamma_w\right\}$$

and $$V_x = -\frac{k}{\gamma_w}\left\{\frac{\partial p}{\partial x}\right\}$$

Thus from equation ③

$$\frac{\partial}{\partial x}\left(-\frac{k}{\gamma_w}\right)\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{k}{\gamma_w}\right)\left\{\frac{\partial p}{\partial y} + \gamma_w\right\} = 0$$

$$\therefore \quad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$

This equation can also be expressed in terms of u or ϕ

where u = excess pressure

ϕ = velocity gradient

$$\nabla^2 \phi = 0$$

$$\nabla^2 u = 0.$$

Relationships between pressures & meaning of Velocity potential

Definition: $$\phi = k\left\{\frac{p}{\gamma_w} + (y - y_0)\right\} = \text{Velocity Potential.}$$

ie ϕ has a similar connotation as in hydraulics

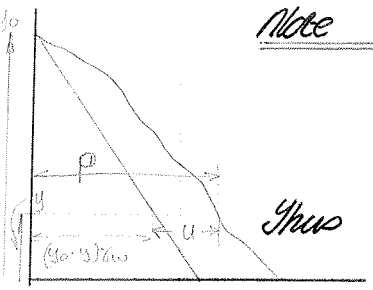
where $h = \frac{v^2}{2g} + \frac{p}{\gamma_w} + y.$

Thus: $$\frac{\partial \phi}{\partial x} = \frac{k}{\gamma_w}\frac{\partial p}{\partial x} = \frac{k}{\gamma_w}\frac{\partial u}{\partial x} = -V_x \quad — ①$$

$u = h\gamma_w$

$P = (y - y_0)\gamma_w$

$i_y = h/\partial y = \frac{\partial u}{\gamma_w \partial y}.$



Note that $p = u - \gamma_w(y - y_0)$

where $u = h\gamma_w$

$$\therefore \quad \frac{\partial p}{\partial x} = \frac{\partial u}{\partial x} = \gamma_w \frac{\partial h}{\partial x}$$

$u = p + \gamma_w(y - y_0)$

$\frac{\partial u}{\partial y} = \frac{\partial p}{\partial y} + \gamma_w.$

Thus $$V_x = k i_x = -\frac{k \partial u}{\gamma_w \partial x} = -k\frac{\partial h}{\partial x}.$$

$$V_y = k i_y = -\frac{k \partial u}{\gamma_w \partial y} = -k\frac{\partial h}{\partial y} = -\frac{k}{\gamma_w}\left\{\frac{\partial p}{\partial y} + \gamma_w\right\}$$

Also

$$\frac{\partial \phi}{\partial y} = k\left\{\frac{1}{\gamma_w}\frac{\partial p}{\partial y} + 1\right\} = \frac{k}{\gamma_w}\left\{\frac{\partial p}{\partial y} + \gamma_w\right\}$$

But $\frac{\partial p}{\partial y} + \gamma_w = \frac{\partial u}{\partial y}$

Thus $\frac{\partial \phi}{\partial y} = \frac{k}{\gamma_w}\frac{\partial u}{\partial y} = -v_y.$

Thus we have $v_y = -\partial\phi/\partial y$

$v_x = -\partial\phi/\partial x.$

Thus from equation (3)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Three dimensional - anisotropic.

The governing equation: $k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial y^2} + k_z \frac{\partial^2 p}{\partial z^2} = 0.$

It is possible to convert this into an equivalent equation with isotropic constants.

Let $\bar{x} = x$

$\bar{y} = \sqrt{\frac{k_x}{k_y}}\, y$

$\bar{z} = \sqrt{\frac{k_x}{k_z}}\, z.$

$\therefore \frac{\partial p}{\partial y} = \frac{\partial p}{\partial \bar{y}} \times \frac{\partial \bar{y}}{\partial y} = \frac{\partial p}{\partial \bar{y}} \sqrt{\frac{k_x}{k_y}}$

$\therefore \frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left\{ \frac{\partial p}{\partial \bar{y}} \sqrt{\frac{k_x}{k_y}} \right\}$

$= \sqrt{\frac{k_x}{k_y}}\, \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial \bar{y}}\right)$

$= \sqrt{\frac{k_x}{k_y}}\, \frac{\partial}{\partial \bar{y}}\left(\frac{\partial p}{\partial \bar{y}}\right) \frac{\partial \bar{y}}{\partial y}$

$\therefore \frac{\partial^2 p}{\partial y^2} = \frac{k_x}{k_y} \frac{\partial^2 p}{\partial \bar{y}^2}$

Similarly $\frac{\partial^2 p}{\partial z^2} = \frac{k_x}{k_z} \frac{\partial^2 p}{\partial \bar{z}^2}.$

Thus:

$$\frac{\partial^2 p}{\partial \bar{z}^2} + \frac{\partial^2 p}{\partial \bar{y}^2} + \frac{\partial^2 p}{\partial x^2} = 0.$$

The boundary conditions must be transferred to the

new coordinate system before the problem can be analysed.

Thus anisotropy (where k is constant in one direction) can be treated.

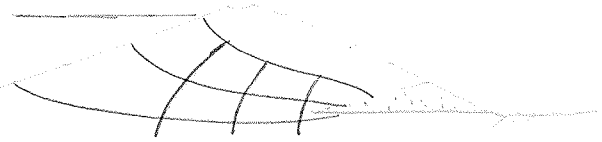
If $\frac{\partial(k_x)}{\partial x} \neq \frac{\partial(k_y)}{\partial y} \neq \frac{\partial(k_z)}{\partial z}$ and $\frac{\partial(k_x)}{\partial x} \neq 0$

then we have non-homogeneity which is far more difficult to deal with.

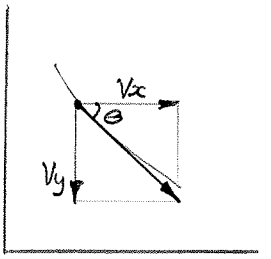
Two dimensional flow - properties of flow & equipotential.

Flow Line

This is the average path taken by a particle of water.



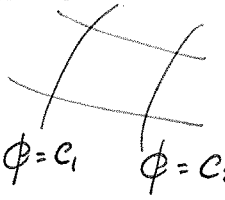
It has two components of velocity. V_x & V_y



$$\tan\theta = \left(\frac{dy}{dx}\right)_{F.L.} = V_y/V_x \quad \text{—— (1)}$$

Equipotential Lines

These are lines of constant velocity potential.



For a particular equipotential

$$\phi(x,y) = C$$

Now $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$ for a particular equipotential.

$$\therefore \quad dy/dx = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

$$= -V_x/V_y \quad \text{—— (2)}$$

Thus from equations (1) & (2)

$$\left(\frac{dy}{dx}\right)_{FL} \times \left(\frac{\partial y}{\partial x}\right)_{EP} = \frac{V_y}{V_x}\left(-\frac{V_x}{V_y}\right) = -1$$

Thus the lines must intersect at 90°.

Consider lines of equal total pressure

$$p = u - \gamma_w(y - y_0) \qquad y_0 \, y$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$$

$$\frac{\partial p}{\partial x} = \frac{\partial y}{\partial x}$$

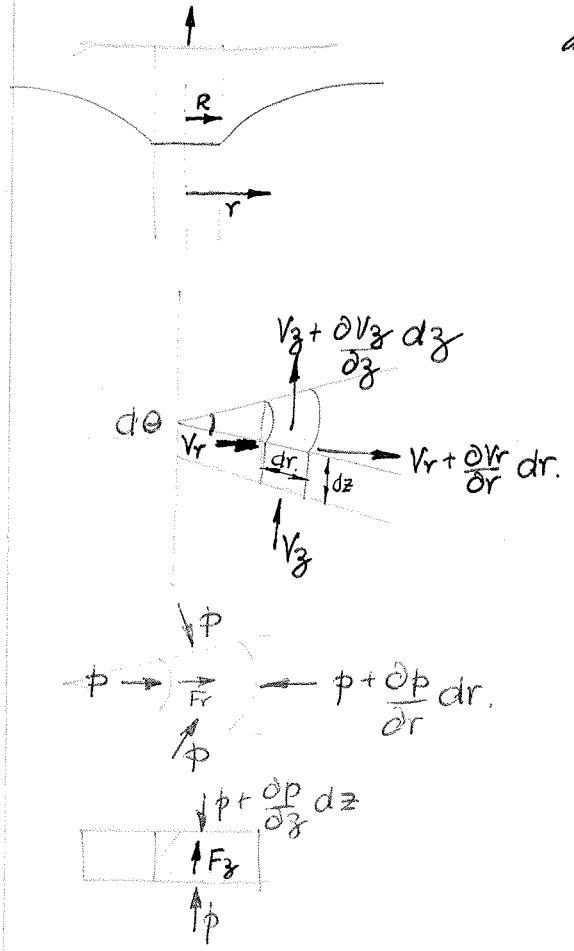
$$\frac{\partial p}{\partial y} = \left\{ \frac{\partial u}{\partial y} - \gamma_w \right\}$$

Now $V_x = \frac{-k}{\gamma_w} \frac{\partial u}{\partial x}$ $\qquad V_y = \frac{-k}{\gamma_w} \frac{\partial u}{\partial y}$

Thus $dy/dx \neq V_x/V_y$

Thus lines of total pressure do not intersect flow lines at 90°. unless $\gamma_w = 0$.

Governing Equations in Polar Coordinates r, ϕ



In all cases of axial symmetry. there is no flow in the angular direction

For equilibrium.

$$\left.\begin{aligned} \frac{\partial p}{\partial r} - Fr &= 0 \\ \partial p/\partial z + \gamma_w - F_z &= 0. \end{aligned}\right\} \quad (1)$$

For continuity

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0 \quad - (2)$$

Darcys law

$$\left.\begin{aligned} Fr &= -\frac{\gamma_w}{k} V_r \\ F_z &= -\frac{\gamma_w}{k} V_z \end{aligned}\right\} \quad (3)$$

Thus : from (1) & (3)

$$\frac{\partial p}{\partial r} + \frac{\gamma_w}{k} V_r = 0 \qquad \therefore V_r = \frac{-k}{\gamma_w} \frac{\partial p}{\partial r}$$

$$\partial p/\partial z + \gamma_w + \gamma_w/k \, V_z = 0 \; . \; \therefore V_z = \frac{-k}{\gamma_w} \left\{ \frac{\partial p}{\partial z} + \gamma_w \right\}$$

Thus substituting in Eq" 2

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0$$

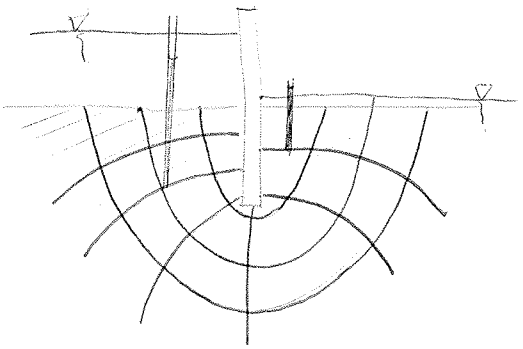
Two classes of problem

1. Confined flow
   all the boundaries of the flow domain are completely defined
2. Unconfined flow
   at least one boundary is a phreatic surface of shape and position unknown.

Methods of Solution

1. Graphical - flow net sketching
2. Exact Analytical - complex variable theory.
   - conformal mapping
3. Approximate Analytical
   - method of fragments
   - Dupuit or Hydraulic theory for unconfined flow
4. Electrical Analog
5. Numerical Analysis
   1. Relaxation method.
   2. Digital computation
6. Viscous flow or Hele - Shaw model.
7. Hydraulic model.

FLOW NET.



Consider that:

N_f = N° of flow paths.
N_d = N° of potential drops

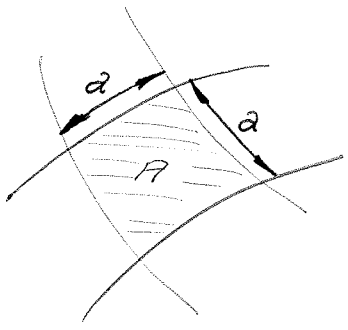
Consider a certain flow path

Flow through $A = \Delta Q$

$$\Delta Q = k \cdot a \cdot \frac{\Delta h}{a}$$

If H = total head loss then

$$\Delta h = \frac{h}{N_d}$$



Thus total flow $Q = \left(k \cdot \frac{h}{N_d}\right) N_f$

$$\therefore Q = Kh \frac{N_f}{N_d}$$

Effect of anisotropy.

In nature $k_H > k_V$ ie $k_x > k_y$

The general equation is

$$k_x \frac{\partial^2 u}{\partial x^2} + k_y \frac{\partial^2 u}{\partial y^2} = 0$$

Let $\bar{y} = \sqrt{\frac{k_x}{k_y}}\, y$ where $\bar{x} = x$.

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y} = \frac{\partial u}{\partial \bar{y}} \sqrt{\frac{k_x}{k_y}}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \bar{y}}\right)\sqrt{\frac{k_x}{k_y}} = \frac{\partial}{\partial \bar{y}}\left(\frac{\partial u}{\partial \bar{y}}\right)\frac{\partial \bar{y}}{\partial y}$$

$$= \frac{k_x}{k_y} \frac{\partial^2 u}{\partial \bar{y}^2}$$

Thus equation becomes $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \bar{y}^2} = 0.$

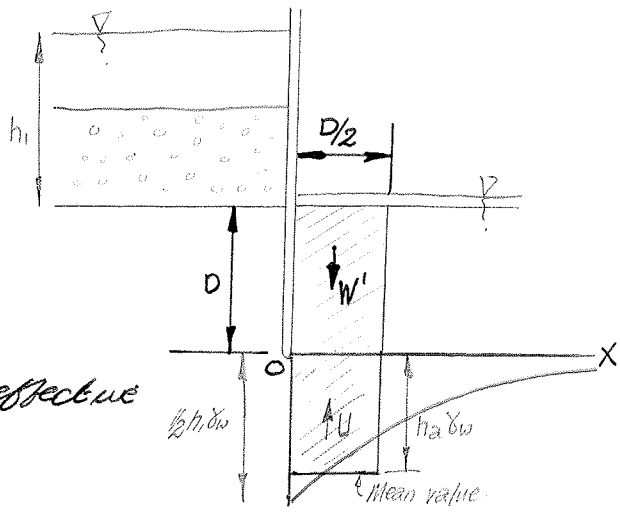
Thus it is possible to perform an ordinary sketch provided that the y scale is distorted by $\sqrt{\frac{k_x}{k_y}}$ ie vertical lines increased in length.

or the x scale is distorted by $\sqrt{\frac{k_y}{k_x}}$ ie horizontal lines become shorter.

The expression for the seepage is then

$$Q = \sqrt{k_x k_y} \cdot \frac{N_f}{N_d} \cdot h.$$

Failure due to heave.



Determine distribution of pressure along ox.

If excess pressure is considered - as obtained from the flow net

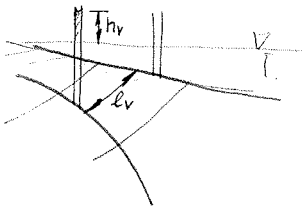
$$U = \tfrac{1}{2} D \cdot \gamma_w \cdot h_2$$

Failure will occur when U = effective weight of soil in prism W'

$$W' = \tfrac{1}{2} D^2 \gamma'$$

Thus $F = \frac{W'}{U} = \frac{D\gamma'}{h_2 \gamma_w}$

Failure due to piping Cohesionless soil.



The exit gradient $i_{exit} = \frac{h_v}{l_v}$

The critical state is when $\sigma' = 0$

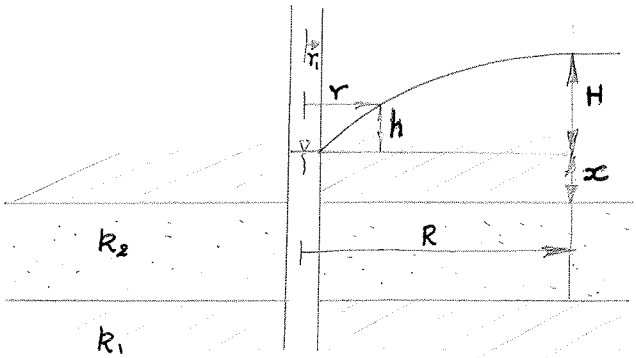
ie $h_{v_c} \gamma_w = l_v \gamma'$

$\therefore i_c = \frac{\gamma'}{\gamma_w}$

It is usually necessary to have F.O.S against piping at least = 3.0.

ANALYTICAL METHODS

One Dimensional.



Flow is in confined layer:

$V_\theta = V_z = 0$

In two dimensional cylindrical:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial z^2} = 0$$

In one dimensional

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0$$

Boundary conditions: $h = 0$ at $r = r_1$

$h = H$ at $r = R$.

Rewrite Eq^n 20. $\frac{1}{r} \cdot \frac{\partial}{\partial r}\left\{ r \frac{\partial h}{\partial r} \right\} = 0$

Integrating this equation twice we have

$$h = a \log_e \frac{r}{b}$$

Condition ①: $0 = \log_e r - \log_e b$ $\therefore b = r_1$.

Condition ② $H = a \log_e \frac{R}{r_1}$.

$$\therefore \frac{h}{H} = \frac{\log_e r/r_1}{\log_e R/r_1}$$ ie h at any radius.

$$\therefore h = H \frac{\log_e r/r_1}{\log_e R/r_1}$$

$$\therefore \frac{dh}{dr} = \frac{H}{\log_e R/r_1} \times \frac{1}{r}$$

Now from Darcys Rule $v = k \frac{dh}{dr}$

For layer thickness = b, Thus $Q = \int_0^{2\pi} k \frac{dh}{dr} \cdot b \cdot r \, d\theta$.

$$= 2\pi \frac{H}{\log_e R/r_1} \cdot bk.$$

Knowing Q, R & H we can calculate k.

Alternate Proof.

Flow out of well = Q

Thus velocity at radius $r = \frac{Q}{2\pi r b}$

But by Darcys law $v_r = k \frac{dh}{dr}$ for a thin layer

$$\therefore \int_{r_1}^{R} \frac{Q}{2\pi r b} dr = \int_0^H k \frac{dh}{dr}$$

$$\therefore Q = \frac{2\pi b k H}{\log_e R/r_1}$$

Two Dimensional Analysis

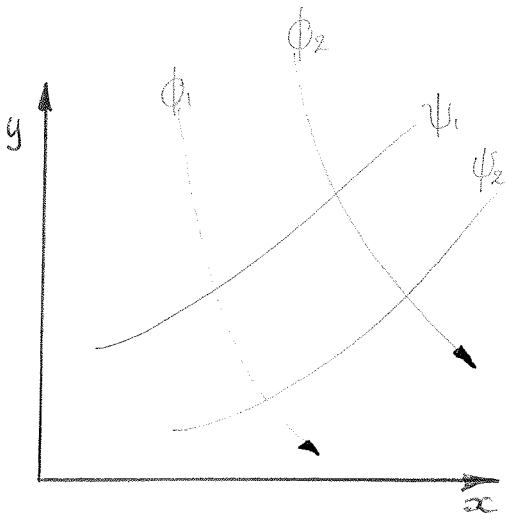
Equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

The problem is to obtain a solution $\phi(x,y)$. The method is to obtain a function which transforms the whole problem into a domain for which the function $\phi(x,y)$ is a simple one

Now let ψ = stream function which satisfies the Laplace equation as well.

ie $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

ie $\psi(x,y)$ and $\phi(x,y)$ are orthogonal.



Let u = velocity in x direction

ie $u = v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ ①

$v = v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ ②

Eq^{ns} ① & ② are the Cauchy Riemen Eq^{ns}

If two functions obey both the Laplace and the Cauchy equations then they are known as Conjugate Harmonic functions. $\{\psi(x,y)\}$ $\{\phi(x,y)\}$.

A combination of such functions may be defined as The Complex Potential : ω

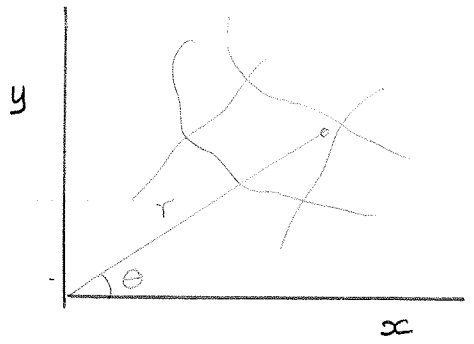
where:

$$\omega = \phi + i\psi$$

This function ω can be operated on by the Laplace operator such that

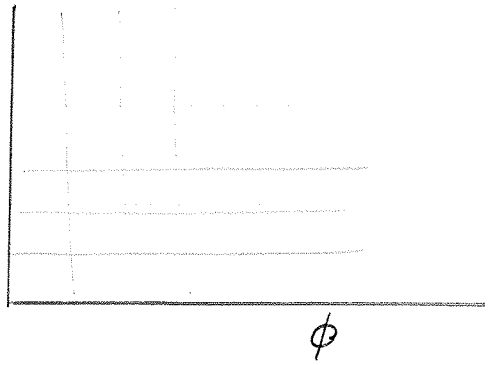
$$\nabla^2 \omega = \nabla^2 \phi + i \nabla^2 \psi = 0$$

Consider two planes.



Z plane

$$z = x + iy$$
$$= re^{i\theta}$$



ω plane

$$\omega = \phi + i\psi.$$

For every value of x & y there is a point on the ω plane. since ϕ & ψ are dependant on x & y.

Lines parallel to the axis are either equipotentials or stream functions in the ω plane. Thus if there is a function

$$\omega = f(z)$$

which will transform curved "Z plane" lines to straight ω plane lines parallel to the axis then the problem is solved.

The function $\omega = f(z)$ will have real and imaginary parts and once

$$\omega = \phi + i\psi$$

we can obtain ϕ & ψ by equating real and imaginary parts.

$\omega = f(z)$ = Mapping function

Conformal mapping

Angular relationship in the physical plane is sustained in the new plane (ω plane)

Example.

Consider $\omega = z^2$ where $z = x + iy$.

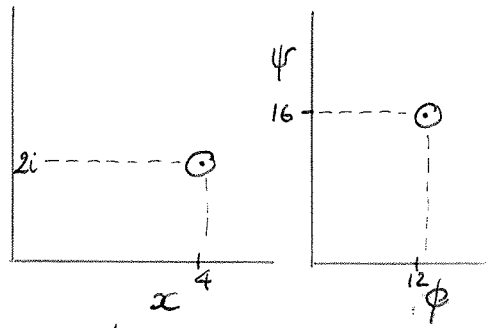
Consider a point in the z plane = $4 + 2i$. What is the corresponding point in the ω plane.

Now $\omega = z^2 = (4+2i)^2 = 16 + 16i - 4.$

$= 12 + 16i$

Thus $\phi = 12$

$\psi = 16.$

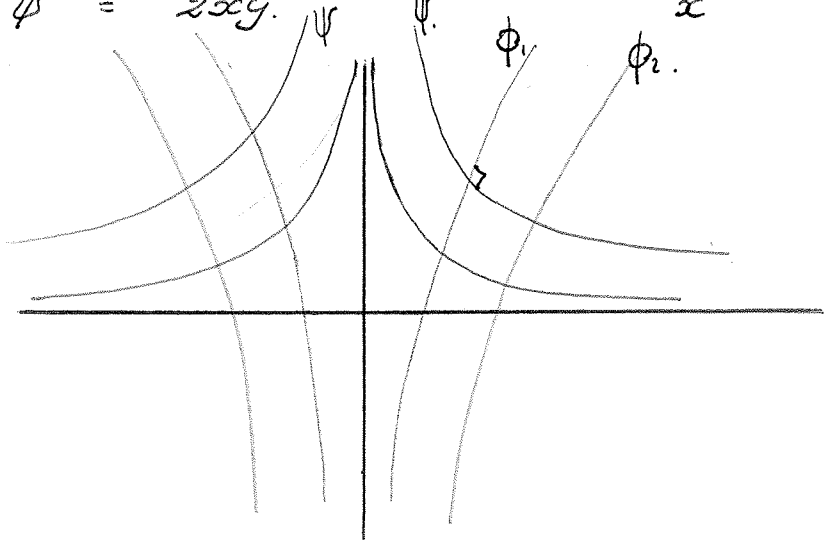


In general with $\omega = (x+iy)^2$

$= x^2 + 2xiy - y^2$

we have $\phi = x^2 - y^2$

$\psi = 2xy.$

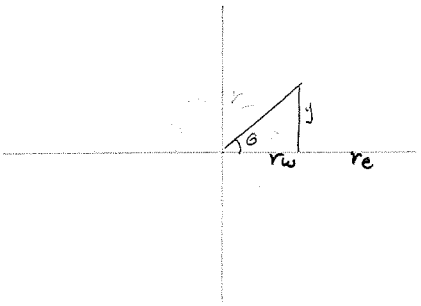


Application:

In most cases where the geometry is "simple" the mapping functions have been determined and are listed in a dictionary of conformal transformations

If the function is not listed then the function can be determined for any polynomial shape using the Schwarz-Christoffel transformation.

Flow into a tunnel.



Determine ϕ such that $\nabla \phi^2 = 0$

for the boundary conditions

1. $r_w^2 < x^2+y^2 < r_e^2$
2. $\phi = \phi_w$ at $x^2+y^2 = r_w^2$
3. $\phi = \phi_e$ at $x^2+y^2 = r_e^2$.

Make use of the log transformation

$$\omega = \frac{Q}{2\pi} \operatorname{Log} z + A + iB = \phi + i\psi$$

But $z = re^{i\theta}$

$$\therefore \operatorname{Log} z = \operatorname{Log} r + i\theta$$

Thus $$\frac{Q}{2\pi}\left\{ \operatorname{Log}(x^2+y^2) + i\theta \right\} + A + iB = \phi + i\psi.$$

$$\therefore \frac{Q}{2\pi}\left\{ \operatorname{Log}(x^2+y^2) + i\tan^{-1} y/x \right\} + A + iB = \phi + i\psi.$$

Thus equating real and imaginary parts.

$$\phi = \frac{Q}{2\pi} \operatorname{Log}(x^2+y^2)^{1/2} + A.$$

$$\psi = \frac{Q}{2\pi} \tan^{-1} y/x + B.$$

Now where $r = r_w$ $\phi = \phi_w$ and $r = re$ $\phi = \phi_e$.

$$\therefore \phi_w = \frac{Q}{2\pi} \operatorname{Log} r_w^2 + A$$

$$\phi_e = \frac{Q}{2\pi} \operatorname{Log} r_e^2 + A.$$

$$\therefore \phi_e - \phi_w = \frac{Q}{2\pi} \operatorname{Log}_e r_e/r_w$$

$$\therefore Q/2\pi = \frac{\phi_e - \phi_w}{\operatorname{Log}_e r_e/r_w} \quad \text{and} \quad A = \phi_w - \frac{Q}{2\pi} \operatorname{Log} r_w$$

Hence

$$\phi = \phi_w + \frac{\phi_e - \phi_w}{\log r_e/r_w} \log r/r_w$$

Thus we have ϕ in terms of x & y and the problem is solved.

The complex velocity.

$$\omega = \phi + i\psi$$

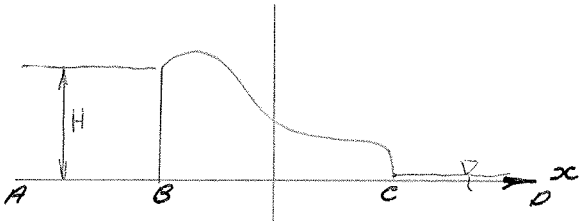
It can be shown that

$$\frac{d\omega}{dx} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

$$W. = u - iv$$

The process is the same as before but velocity is used instead of potentials. This is very useful in solving problems of unconfined flow.

Application



Along AB & CD : $u = 0$
Along BC : $v = 0$

We have to find $W = u - iv$.

From the boundary conditions we know

1/ Along BC $v = 0$
$\therefore$ W must be real. for $-b < x < b$.

2/ Along AB & CD $u = 0$
$\therefore$ W is imaginary. for $|x| > b$

Consider the function $f(z) = \sqrt{b - z}$ along $y = 0$
if $z < b$ then $f(z)$ is real.
if $z > b$ then $f(z)$ imaginary.
This thus satisfies the properties for +ve x values.
similarly the function $f(z) = \sqrt{b+z}$.
real for $z > -b$ = real.
for $z < -b$ = imaginary

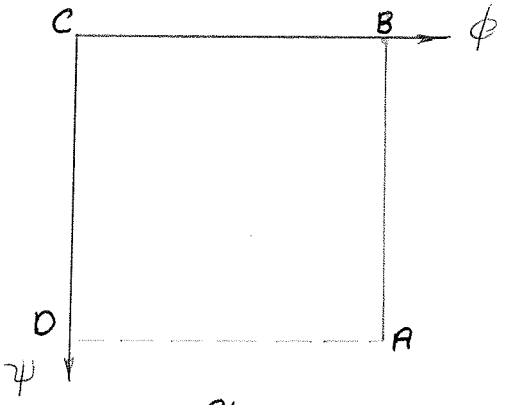
Thus the function $\sqrt{b^2 - z^2}$ is real between $-b$ and b
and imaginary outside. for $y = 0$

However we have that as $y \to \infty$ u and $v \to 0$

Thus $W = \frac{dw}{dz} = \frac{M}{\sqrt{b^2 - z^2}}$

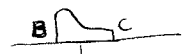
Thus integrate w.r.t z to obtain $w = M \sin^{-1} \frac{z}{b} + N$.

$= \phi + i\psi$.



Now BC = flow line ie ψ = constant
AB = Equipotential ϕ = const = H
CD = " ψ = 0

At C : $\phi = \psi = 0$
At B : $\phi = H$
$\psi = 0$.



Point where the two meet both must be zero

w Plane.

We thus have the boundary conditions

At C. $w = 0$ and $z = +b$.

At B $w = H$ and $z = -b$.

We can write $w = M \sin^{-1} z/b + N$

and solving for the boundary conditions

$$w = \frac{-H}{\pi} \cos^{-1} z/b.$$

$$\therefore \quad z = -b \cos \frac{\pi w}{H}$$

$\cos(A+B)$
$= \cos A \cos B - \sin A \sin B$
$\sin ix = i \sinh x$.!
$\cos ix = \cosh x$

$$\therefore \quad x + iy = -b \cos \pi/H \, (\phi + i\psi)$$

$$\therefore \quad x + iy = -b \left\{ \cos \frac{\pi}{H}\phi \cos \frac{\pi}{H} i\psi - \sin \frac{\pi}{H}\phi \sin \frac{\pi}{H} i\psi \right\}$$

$$= -b \left\{ \cos \frac{\pi}{H}\phi \cosh \frac{\pi}{H}\psi - i \sin \frac{\pi}{H}\phi \sinh \frac{\pi}{H}\psi \right\}$$

Thus $x = -b \cos \frac{\phi\pi}{H} \cosh \frac{\pi\psi}{H}$

$y = b \sin \frac{\pi\phi}{H} \sinh \frac{\pi\psi}{H}$

Thus $\cos \frac{\pi\phi}{H} = x / b \cosh \frac{\pi\psi}{H}$ and $\sin \frac{\pi\phi}{H} = y / b \sinh \frac{\pi\psi}{H}$.

But $\sin^2 + \cos^2 = 1$ $\cosh^2 - \sin^2 = 1$

Thus squaring and adding gives the answer

of ψ in t.o x, y

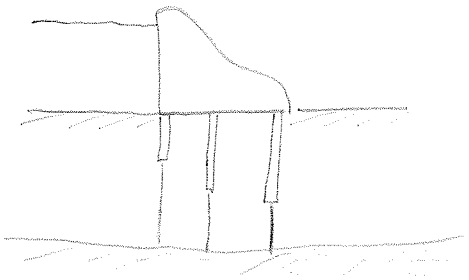
ϕ i.t.o x, y.

Approximate analytical techniques

a: Method of fragments. Pavlovsky.

This method is good for obtaining the discharge but is not so good for pore pressures.

Through all the difficult points one considers the equipotentials to be vertical



Theory.

Consider that the discharge in the nth fragment is known

$$\therefore q = \frac{k h_n}{\Phi_n}$$ where Φ_n = shape factor of the nth fragment.

Now the discharge must be the same for all fragments.

$$\therefore q = \frac{k h_1}{\Phi_1} = \frac{k h_2}{\Phi_2} = \dots\dots = \frac{k h_n}{\Phi_n} = Q$$

and the total head loss = $H = \sum_1^n h_n$.

$$\therefore q = \frac{k \sum_1^n h_n}{\sum_1^n \Phi_n} = \frac{kH}{\sum_1^n \Phi_n}$$

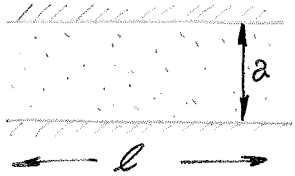
Thus if we know the shape factors of different segments and we know k and H we can determine q.

Also the head loss in any segment

$$h_n = \frac{\Phi_n}{\sum_1^n \Phi_n} H.$$

Examples of shape factors.

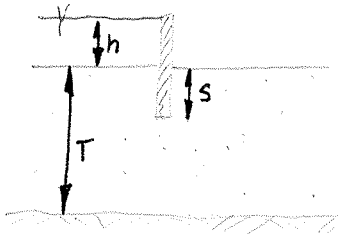
1. Layer



1/ $q = k a \frac{b}{l} = \frac{k h}{S}$

Thus $\Phi = \frac{l}{a}$.

2 Cutoff.



2/ $q = \frac{k h K'}{2K}$

Where $K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m^2 \sin^2\phi}}$

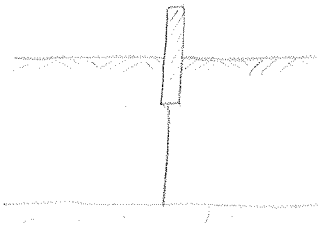
and m = modulus $= \sin \frac{\pi s}{2T}$

and where K' = same function only with m' instead of m

$m'^2 = 1 - m^2$

NOTE: s, T are known. hence calculate m. & m' From tables one can then read K & K' and hence calculate $\bar{\phi}$

3. Left hand side of cut-off

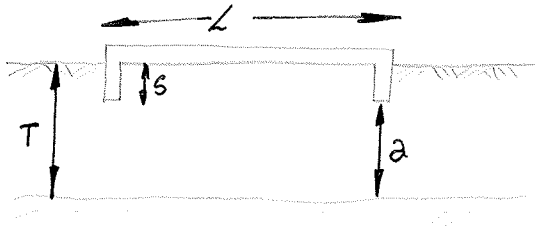


$$q = \frac{kH}{2} \times \frac{\bar{\phi}}{I_0}$$

Thus $\bar{\phi}$ is half the previous value

$$\therefore \quad \bar{\phi} = \frac{K}{K'}$$

4. Portion between two cut-offs.



If $L \leq 2s$ then

$$\bar{\phi} = 2 \log (1 + L/2a)$$

If $L \geq 2s$

$$\bar{\phi} = 2 \log (1 + s/2) + \frac{L - 2s}{T}$$

Example solved using above shape factors.

Fragments ① & ③

$$\bar{\phi} = \frac{K}{K'}$$

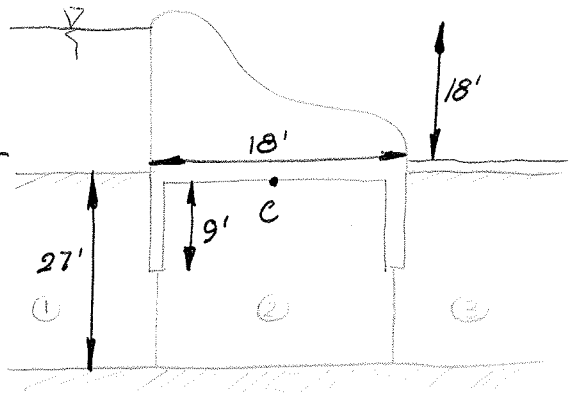
Now $m = \sin\left(\frac{\pi s}{2T}\right) = \sin \frac{9\pi}{2 \times 27} = \sin \frac{\pi}{6}$

$\therefore m = 1/2$

and $m' = 1 - 1/4 = 3/4$

Hence from tables of K vs m or m'

$K/K' = 0.782.$



Fragment N°2.

$L = 2s \quad \therefore \quad \bar{\phi} = 2 \log \left(1 + \frac{18}{36}\right) = 2 \log_e 1.5$

$= 0.811$

Thus $\sum_1^3 \bar{\phi}_n = 2 \times 0.782 + 0.811 = 2.375.$

Thus $q = \frac{k\ 18}{2.325}$

In order to obtain the efficiency one can consider the flow with no cut-offs

ie approximately flow through a tube.

$$q = \frac{k\,h}{L}\,a = \frac{k \cdot 18}{18/27}$$

Thus % of cut-off $\frac{Q_{cut}}{Q_{no\ cut}} = \frac{k\ 18}{2.375} \Big/ k \cdot 27 \times 100\%$

$= 28\%$

Head losses.

Head loss $h_n = \frac{H\,\Phi_n}{\Sigma \Phi_n}$.

Thus in ① $h_1 = \frac{18 \times 0.782}{2.375} = 5.9'$

Thus head loss in ② $= 18 - 11.8$

$= 6.2'$

Thus head loss up to the point C $= 5.9 + \frac{6.2}{2} = 9.0$

Note:

The head loss in ⑤ is over a flow path length of $L + 2S$.

$\therefore$ Head loss rate $= \frac{6.2}{36}$ ft/ft.

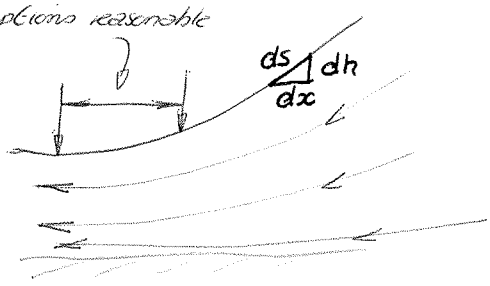
Dupuit Theory for unconfined flow

The problem is to solve the top flow line

Two assumptions are made.

a. For small inclinations of the line of seepage the flow is horizontal.

b. The hydraulic gradient is equal to the slope of the free surface and is invariant with depth.



The slope $= \frac{dh}{dx}$

whereas the correct $i = \frac{dh}{ds}$

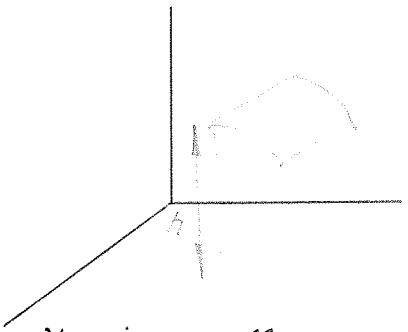
However for small inclinations of the phreatic surface these assumptions give reasonable answers.

The error involved in the Dupuit assumptions

In general $\phi = f(x\,y\,z) = -kh(x,y,z) = -k\left\{\frac{p}{\gamma_\omega} + z\right\}$

For a free surface $\phi = 0$

$\therefore \quad h(x\,y\,z) = \bar{z}$ where $\bar{z}$ is the mean height of the free surface



Expand $h(x\,y\,z)$ about $\bar{z}$

$$h(x\,y\,z) = h(x\,y\,\bar{z}) + \frac{\partial h}{\partial z}_{(z\to\bar{z})}(z-\bar{z}) + \cdots$$

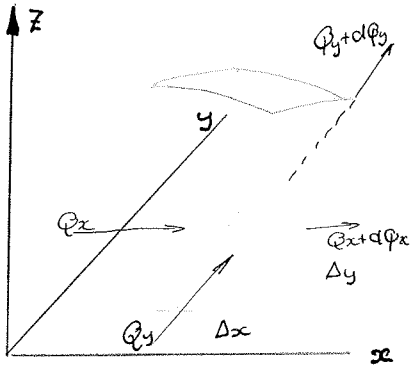
If the free surface is relatively level then $z-\bar{z}$ is small and $\frac{\partial h}{\partial z}$ is small.

Thus $h(x\,y\,z) = h(x\,y\,\bar{z})$

but $\bar{z}$ is a constant

Thus $h = f(x,y)$

and $h(x\,y) = z$.



$$Qx = -k\,h\,\frac{\partial h}{\partial x}\,\Delta y.$$

$$Qx + dQx = -kh\,\frac{\partial h}{\partial x}\Delta y + \frac{\partial Q}{\partial x}\Delta x$$

$$= -kh\,\frac{\partial h}{\partial x}\Delta y + \frac{\partial}{\partial x}\left\{-kh\,\frac{\partial h}{\partial x}\Delta y\right\}\Delta x$$

$$\therefore \quad dQx = -\Delta y\,\Delta x\,k\,\frac{\partial}{\partial x}\left\{h\,\frac{\partial h}{\partial x}\right\}$$

There is no flow in the z direction by definition

Similarly $dQy = -\Delta y\,\Delta x\,k\,\frac{\partial}{\partial y}\left\{h\,\frac{\partial h}{\partial y}\right\}$

Now for uniform flow $Qx + dQx \neq Qy + dQy = Qx + Qy$

$$\therefore \quad dQx + dQy = 0$$

Thus $$\frac{\partial}{\partial x}\left\{h\,\frac{\partial h}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{h\,\frac{\partial h}{\partial y}\right\} = 0$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{\frac{1}{2}\frac{\partial}{\partial x}(h^2)\right\} + \frac{\partial}{\partial y}\left\{\frac{1}{2}\frac{\partial}{\partial y}(h^2)\right\} = 0$$

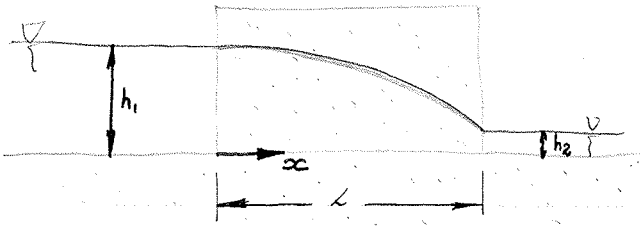
$$\therefore \quad \frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0$$

This equation is the basis of the Casagrande sketching

If there is no flow in the y direction then the equation becomes.

$$\frac{\partial^2 h^2}{\partial x^2} = 0$$

Application



$$\frac{\partial^2 h^2}{\partial x^2} = 0$$

$$\therefore \iint \frac{\partial^2 h^2}{\partial x^2} = h^2 = Ax + B$$

We have the boundary conditions.

When $x = 0 \quad h = h_1$

$x = L \quad h = h_2$

Thus $h_1^2 = B$

and $h_2^2 = AL + h_1^2 \quad \therefore A = \frac{h_2^2 - h_1^2}{L}$.

Thus $h = \sqrt{h_1^2 - \left(\frac{h_1^2 - h_2^2}{L}\right)x}$. $\quad \frac{2h\,dh}{dx} = -\frac{(h_1^2 - h_2^2)}{L}$

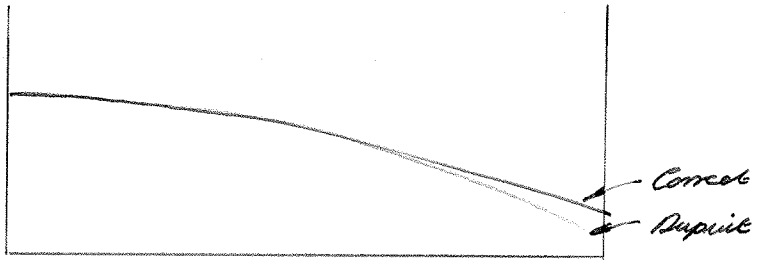
This is the Dupuit parabola for the phreatic surface.

Now $q = -kh\frac{dh}{dx}$ /unit depth. by Dupuits second assumption.

$$= -kh\left\{\frac{d}{2h}\left(-\frac{(h_1^2 - h_2^2)}{L}\right)\right\}$$

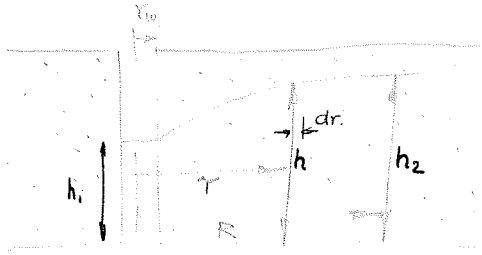
$$q = k\,\frac{h_1^2 - h_2^2}{2L}$$

As h_2 becomes smaller so the error in the flow here becomes greater. However the discharge remains empirically correct.



Thus close to the outlet boundary the predicted pressures will be wrong.

Seepage into a well



Assume that flow lines are horizontal ie flow through any cylinder is the same

The main assumption is that $i = \frac{dh}{dr}$ and does not vary with depth

Flow out of well = Q.

$\therefore$ Velocity at radius r = $\frac{Q}{2\pi r h}$.

But by darcy's law and Dupuits assumption

$$v_r = k\frac{dh}{dr}$$

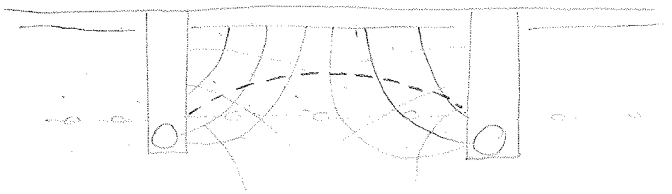
Thus $$\frac{Q}{2\pi r h} = k\frac{dh}{dr}$$

$$\therefore \frac{Q}{2\pi k}\int_{r_w}^{r_1}\frac{1}{r}\,dr = \int_{h_1}^{h_2} h\,dh$$

$$\therefore \frac{Q}{2\pi k}\log_e r_1/r_w = \frac{h_2^2 - h_1^2}{2}$$

$$\therefore Q = \frac{\pi k\,(h_2^2 - h_1^2)}{\log_e r_1/r_w}.$$

Drainage of landslides

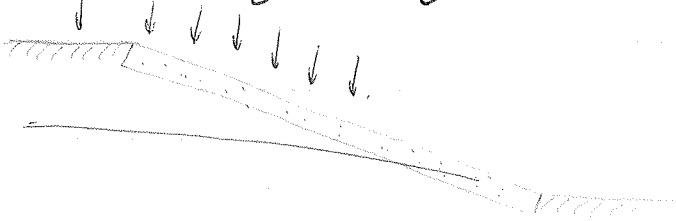


Under severe rainfall conditions it is considered that the water level can rise to the ground surface

The drains have the effect of altering the pore pressure on a average slip plane because of the altered flow pattern.

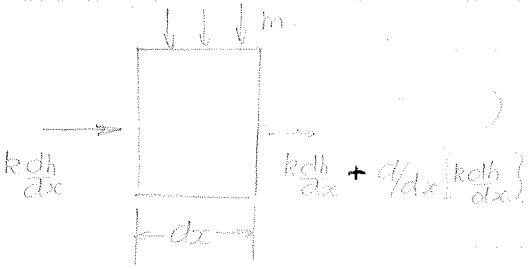
In order to solve the actual three dimensional problem information is required as to how much of the water in the slope is derived from rainfall and how much flows in from the ground behind

In order to analyse the problem roughly one can make use of the Dupuit assumptions. This is reasonable since because the slopes are fairly gentle the inclination of the piezometric surface is gentle.



Consider that there is a uniform infiltration quantity of M / unit area

ie an infiltration velocity of m inches/year.



By continuity of plane flow we must have.

$$k \cdot \frac{d}{dx}\left\{ h \cdot \frac{dh}{dx} \right\} + m = 0$$

Thus integrating once

$$\frac{dh}{dx} k h^2 + mx = C$$

twice

$$\frac{kh^2}{2} + \frac{mx^2}{2} = C_1 x + C_2.$$

$$kh^2 + mx^2 = Bx + B_2.$$

$x = 0 \quad h = 0 \quad \therefore B_2 = 0$

$x = L \quad h = 0 \quad B = 0$

$$\therefore kh^2 + mx^2 = mLx$$

$$\therefore kh^2 = mLx - mx^2$$

$$= mx(L - x)$$

$$\therefore h = \sqrt{\frac{m}{k} x(L-x)}.$$

Thus h_{max} is at $x = L/2$.

$$\therefore h_{max} = \sqrt{\frac{m}{k} \frac{L^2}{4}} \quad \therefore \frac{h_m}{L} = \frac{1}{2}\sqrt{\frac{m}{k}}$$

h_{max} : is known for a required factor of safety.
k : is known.
If m is known then can calculate L required.

eg $m = k/10$ let $h_m = 5'$

$$\therefore \frac{5}{L} = \frac{1}{2}\sqrt{0.1} = 0.16$$

$$\therefore L = 5/0.16 = 31'$$

Electrical analogues.

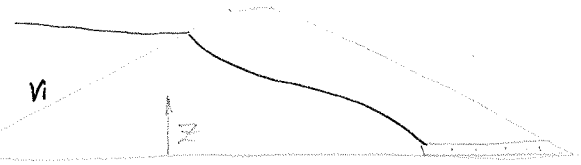
Total head h or ϕ	=	Voltage V
Permeability k	=	Conductivity σ
Discharge vel v	=	Current I.
$v = -k\frac{\partial h}{\partial x}$	=	$I = -\sigma\frac{\partial V}{\partial x}$

Equipotential $h = C$ (note has $a = \delta h$.) : $V = C$

Impervious $\frac{\partial h}{\partial x} = 0$ $\qquad$ $\frac{\partial V}{\partial x} = 0$.

To obtain stream lines one must invert the pattern used for obtaining the equipotentials. This is true in principle but in practice may not always work.

Unconfined flow.

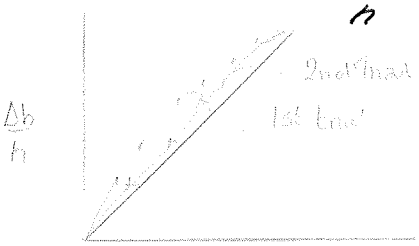


1/ We know V_i

2/ Know that for the top flow line ϕ or h is linear w.r.t z.

Thus perform a trial cut and measure $\frac{\Delta h}{h}$

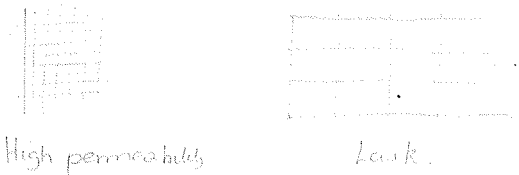
The results are plotted as shown and thus a further small amount may be trimmed off.



The electrical analogues are not good for detail but can be used for quite complicated systems using conducting inks.

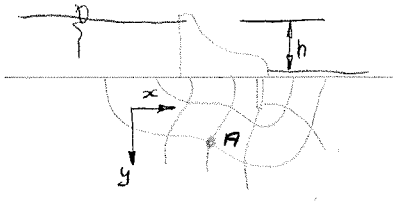
One can model a domain of an equivalent conductivity by using different line thicknesses or spacings

A well can be modeled using a thick line and a cut-off merely by cutting lines.



Numerical Methods.

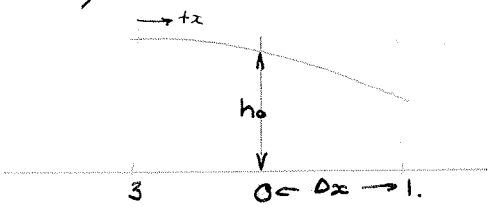
The process is to convert the partial differential equation into a set of simultaneous linear equations.

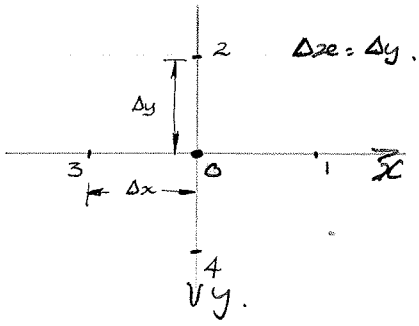


At point A we have the equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Consider the variation of h along the x plane.





The variation of h about h_0 in the x direction can be obtained using Taylors theorem for l close to 0.

$$h_1 = h_0 + \Delta x \left(\frac{\partial h}{\partial x}\right)_0 + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 h}{\partial x^2}\right)_0 + \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 h}{\partial x^3}\right)_0 + \cdots$$

$$h_3 = h_0 - \Delta x \left(\frac{\partial h}{\partial x}\right)_0 + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 h}{\partial x^2}\right)_0 - \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 h}{\partial x^3}\right)_0 + \cdots$$

Adding

$$h_1 + h_3 = 2h_0 + (\Delta x)^2\left\{\frac{\partial^2 h}{\partial x^2}\right\}_0 + \frac{2(\Delta x)^4}{4!}\left\{\frac{\partial^4 h}{\partial x^4}\right\}_0.$$

$$\therefore \quad \frac{h_1 + h_3 - 2h_0}{(\Delta x)^2} = \frac{\partial^2 h}{\partial x^2}.$$

As the mesh gets small so Δx is smaller and the accuracy is better. If h changes very rapidly there will be errors due to neglecting some of the terms.

We thus have

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_1 + h_3 - 2h_0}{(\Delta x)^2}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{h_2 + h_4 - 2h_0}{(\Delta y)^2}.$$

Thus for $\Delta x = \Delta y$ the finite element form of the Laplace equation becomes.

$$h_1 + h_2 + h_3 + h_4 - 4h_0 = 0.$$

We have to have known values at a certain number of mesh points before the solution can be obtained ie boundary conditions known.

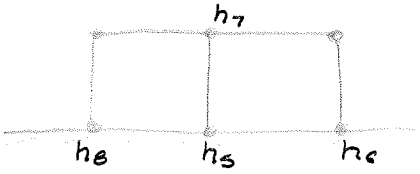
Relaxation Methods of Solution

We have a set of simultaneous linear equations which must be solved at each nodal point

Equations

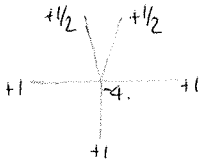
a. Central node $h_1 + h_2 + h_3 + h_4 - 4h_0 = 0.$

b. Boundary node.



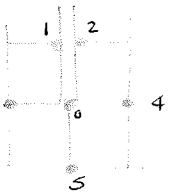
$$h_7 + \frac{h_8}{2} + \frac{h_6}{2} - 2h_5 = 0.$$

ie $2h_8 + 2h_6 + h_7 - 4h_0 = 0.$



c. Sheet pile node.

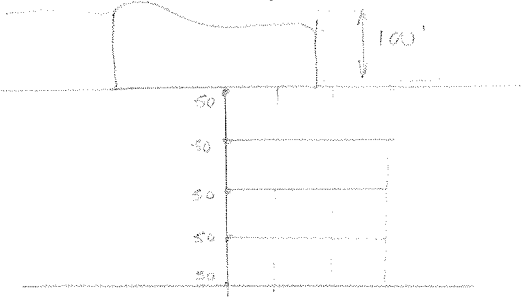
$$\tfrac{1}{2}h_1 + \tfrac{1}{2}h_2 + h_3 + h_4 + h_5 - 4h_0 = 0.$$



Method.

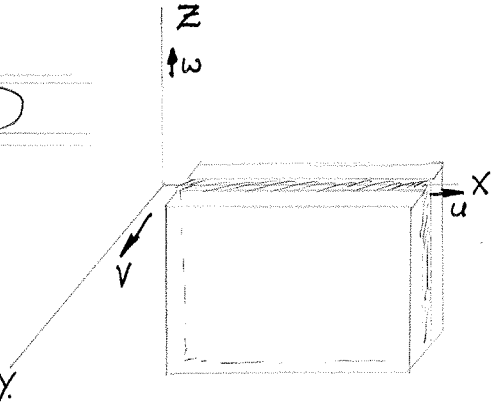
Draw a rough flow net to obtain approximate nodal point values.

Then in a tabular form calculate the residuals and make corrections to the nodal point values until the residuals are reduced to an acceptable value.



The Hele-Shaw model.

The analog is the viscous flow between parallel plates



u, v, w : fluid velocities

ρ_g : density of fluid

μ : viscosity

Assumptions

1. $V = 0$.
2. Since viscous terms are large

one can neglect inertia

3. Second derivatives of the velocity components in the X & Z directions may be neglected with respect to $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 \omega}{\partial y^2}$ This is not true if an obstruction is placed in the field. Thus the method is not good for detail.

Navier - Stokes.

$$1/ \quad \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0$$

$$2/ \quad -\partial p/\partial y = 0$$

$$3/ \quad \mu \frac{\partial^2 \omega}{\partial y^2} - \frac{\partial p}{\partial z} - \rho y = 0$$

$u = \omega = 0$ if $y = \pm b/2$

Integrating these equations and putting in the boundary conditions we get.

$$\omega = \frac{1}{2\mu}\left(y^2 - \frac{b^2}{4}\right)\frac{\partial}{\partial z}(p + \rho g z)$$

$$u = \frac{1}{2\mu}\left(y^2 - \frac{b^2}{4}\right)\frac{\partial}{\partial x}(p + \rho g z).$$

This solution is for the velocity distribution

Hence $$u_{ave} = \int_{-b/2}^{+b/2} u\,dy \Big/ b.$$

$$= -\frac{b^2 \gamma}{12\mu}\frac{\partial}{\partial x}(z + p/\gamma). \qquad (\rho g = \gamma).$$

$$\omega_{ave} = \int_{-b/2}^{+b/2} \omega\,dy \Big/ b$$

$$= -\frac{b^2 \gamma}{12\mu}\frac{\partial}{\partial z}(z + p/\gamma)$$

Thus we can write these equations in the form

$$u_{ave} = -k\frac{\partial h}{\partial x} \qquad h = (z + p/\gamma)$$

$$\omega_{ave} = -k\frac{\partial h}{\partial z} \qquad k = \frac{b^2 \gamma}{12\mu}.$$

The system is thus an "exact" analog. The plates must be parallel otherwise there is an effective variation in the permeability

Problems involving flow through porous media

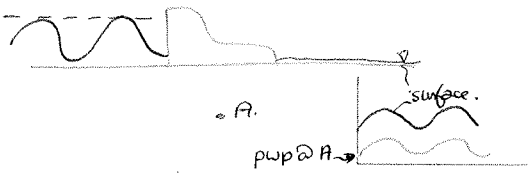
a: Underseepage and drainage problems. : Vol 1 p81 9th I.C.O.S.M.F.E. Conf.

b: Pumping tests and ground water lowering
Mansur & Dietrich SM4 July 65

c: Unsteady state Abbot & Gerge Vol 1 Paris Conf.

In unsteady state problems we have the boundary conditions changing with time

i/ Confined system.

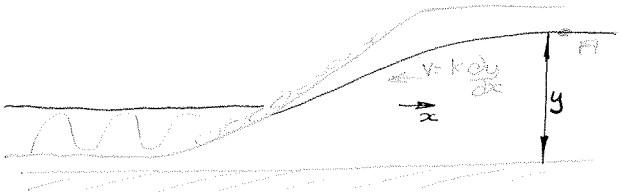


The response for practical purposes is instantaneous. One need just write as a boundary condition the variation in head with time

The pore water pressure may however lag because of

a. Compressible fluid
b. Compressible soil skeleton.
c. Compressible measuring device.

ii/ Unconfined system.

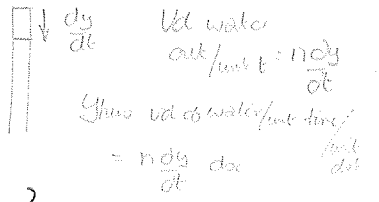


$$q = ky\frac{dy}{dx} \quad \text{by Dupuit.}$$

$$\frac{dq}{dx} = k\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\}$$

The change in quantity of water flowing in the x direction

$$dq = n\frac{dy}{dt}\,dx$$



$$dy/dt = \frac{1}{n}\frac{dq}{dx}$$

$$= \frac{k}{n}\left\{\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right\}$$

Assume that $\left(\frac{dy}{dx}\right)^2 \ll y\frac{d^2y}{dx^2}$.

$$\therefore \frac{dy}{dt} = \frac{ky}{n}\frac{d^2y}{dx^2}$$

$$\therefore \frac{dy}{dt} = \frac{kh}{n}\frac{d^2y}{dx^2}$$

The point is that one ends up with a consolidation equation which can be solved for the appropriate boundary conditions.